Math 255A' Lecture 3 Notes

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1 Brief Introduction to Banach Spaces

1.1 Seminorms and norms

We will denote X as a vector space over $\mathbb{F}$.

Definition 1.1. A **seminorm** on X is a function $p : X \to [0, \infty)$ such that

1. $p(x + y) \leq p(x) + p(y)$
2. $p(\lambda x) = |\lambda| p(x)$ for all $x \in X, \lambda \in F$.

We call p a **norm** if $p(x) = 0 \implies x = 0$ (coercivity of p).

Remark 1.1. The second property implies $p(0) = 0$.

A norm has an associated metric $d(x, y) = p(x - y)$.

Definition 1.2. If p is a norm, the pair (X, p) is called a **normed space**. If X is complete with respect to this metric, we call it a **Banach space**.

Proposition 1.1. *In a normed space, addition and scalar multiplication are continuous.*

Lemma 1.1. *Let p, q be seminorms on X. The following are equivalent:*

1. $p \leq q$
2. $\{x \in X : q(x) \leq 1\} \subseteq \{x \in X : p(x) \leq 1\}$
3. $\{x \in X : q(x) < 1\} \subseteq \{x \in X : p(x) < 1\}$
4. $\{x \in X : q(x) < 1\} \subseteq \{x \in X : p(x) \leq 1\}$

Proof. (4) $\implies$ (1): :et $x \in X$ be such that $q(x) \leq a$. Let $\varepsilon > 0$ be arbitrary. Then

$$q\left(\frac{x}{a + \varepsilon}\right) \leq \frac{a}{a + \varepsilon} < 1,$$

so $p(x/(a + \varepsilon)) \leq 1$. This implies $p(x) \leq a + \varepsilon$. □

Proposition 1.2. *For all $x, y \in X$, $|p(x) - p(y)| \leq p(x - y)$.*

Proof. The triangle inequality gives $p(x) \leq p(y) + p(x-y)$, so $p(x) - p(y) \leq p(x-y)$. Flip x and y to get the negative version. □

Remark 1.2. This tells us that the norm in a normed space is Lipschitz.

Definition 1.3. Two norms are **equivalent** if they generate the same topology.

Proposition 1.3. *$\|\cdot\|_1$ and $\|\cdot\|_2$ on X are equivalent if and only if there are constants $c, C > 0$ such that*

$$c\|\cdot\|_1 \leq \|\cdot\|_2 \leq C\|\cdot\|_1.$$

Proof. ($\Longleftarrow$): Given these inequalities, consider $B^2(x, \varepsilon) = \{y \in X : \|y - x\|_2 < \varepsilon\}$. This contains $B^1(x, \varepsilon/c)$. So the topology $\mathcal{T}_{\|\cdot\|_2}$ contains $\mathcal{T}_{\|\cdot\|_1}$. The other inequality gives the reverse inclusion.

($\Longrightarrow$): Assume $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent. Consider $B^1(0,1)$. It must contain some $\|\cdot\|_2$ open neighborhood U of 0. So there is some $\varepsilon > 0$ such that $B^1(0,1) \supseteq B^2(0, \varepsilon)$. This tells you that $\|\cdot\|_1 \leq (1/\varepsilon)\|\cdot\|_2$ by the lemma. We can do the reverse to get another inequality. □

Definition 1.4. $(X, \|\cdot\|)$ and $(X', \|\cdot\|')$ are **isometric**[1] if there is a linear bijection $A : X \to X'$ such that $\|Ax\|' = \|x\|$ for all $x \in X$. They are **isomorphic** if $\|\cdot\|$ and $\|A(\cdot)\|'$ are equivalent.

1.2 Examples of Banach spaces

Example 1.1. Let X be a Hausdorff[2] topological space. Then let the space $C_b(X) = \{\text{bounded continuous functions } X \to \mathbb{F}\}$ equipped with the **uniform/sup norm** $\|f\| := \sup_{x \in X} |f(x)|$. Then $(C_b(X).\|\cdot\|)$ is a Banach space.

Example 1.2. If I is any set with the discrete topology, the previous example gives $C_b(I) = \ell^\infty(I) = \{(x_i)_{i \in I} \in \mathbb{F}^I : \sup_i |x_i| < \infty\}$. If $I = \mathbb{N}$, we call $\ell^\infty(\mathbb{N}) = \ell^\infty$.

Example 1.3. If X is locally compact, $C_0(X) = \{f \in C_b(X) : \forall \varepsilon > 0, \{|f| \geq \varepsilon\} \text{ is compact}\}$ is a closed subspace of $C_b(X)$. If X is compact, $C_b(X) = C_0(X) =: C(X)$.

We call $c_0 = C_0(\mathbb{N}) = \{(x_i)_i \in \mathbb{F}^{\mathbb{N}} : x_i \xrightarrow{i \to \infty} 0\}$.

Example 1.4. Let $(X.\Sigma, \mu)$ be a measure space. Then $L^p(\mu)$ for $1 \leq p \leq \infty$ is a Banach space with the norm $\|f\|_p = (\int |f|^p)^{1/p}$ if $p < \infty$ and $\|f\|_\infty = \operatorname{ess\,sup} |f|$.

[1] This is generally a really rigid condition. Theorems about isometry are almost always easy or false.

[2] We don't actually need this, but analysts don't like thinking about non-Hausdorff psaces. If you ever wonder why the Hausdorff condition is there in a situation, it might be sociological prejudice.

Example 1.5. Fix $n \geq 1$, and let $C^{(n)}([0,1]) = \{f : [0,1] \to \mathbb{F} \text{ with } n\text{-fold conts. derivs.}\}$. With the norm $\|f\| = \max_{-\leq k \leq n} \sup_x |f^{(k)}(x)|$, $C^{(n)}([0,1])$ is a Banach space.

Similar spaces called **Sobolev spaces**, where we do not require the last derivative to be continuous. These are useful for PDEs; people will define a Banach space of functions with the correct amount of regularity to find a solution to a PDE inside.

1.3 Bounded linear operators

Definition 1.5. A **continuous linear operator** $X \to X'$ is a linear operator which is continuous according to the norm topologies.

Proposition 1.4. *Let $T : X \to X'$ be linear. The following are equivalent:*

1. *T is continuous.*
2. *T is continuous at 0 ($= 0_X$).*
3. *T is continuous at some point in X.*
4. *There exists some $c < \infty$ such that $\|Tx\|' \leq c\|x\|$ for all $x \in X$.*

The proof is similar to the proof of the lemma from before. Because of condition 4, continuous linear operators are often referred to as **bounded**.

Definition 1.6. $\mathcal{B}(X, X')$ denotes the vector space of bounded linear operators $X \to X'$. This has the **operator norm**

$$\begin{aligned} \|T\| &= \inf\{c > 0 : \|Tx\|' \leq c\|x\| \ \forall x \in X\} \\ &= \sup\left\{\frac{\|Tx\|'}{\|x\|} : x \in X \setminus \{0\}\right\} \\ &= \sup\{\|Tx\|' : \|x\| = 1\}. \end{aligned}$$

Example 1.6. Fix a measure space (X, Σ, μ) and $1 \leq p \leq \infty$, and let $\varphi \in C^\infty(\mu)$. Then the **multiplication operator** $M_\varphi : L^p(\mu) \to L^p(\mu)$ sending $f \mapsto \varphi f$ is bounded:

$$\|M_\varphi f\|_p = \left(\int |\varphi f|^p\right)^{1/p} \leq \|\varphi\|_\infty \|f\|_p.$$

We can choose a positive measure set where φ is close to its essential supremum and let f be the indicator of that set. This makes $\|M_\varphi f\|_p$ arbitrarily close to $\|\varphi\|_\infty \|f\|_p$, so we get $\|M_\varphi\| = \|\varphi\|_\infty$.

Example 1.7. Consider $L^p(\mu)$. Assume $K : X \times X \to \mathbb{F}$ is such that there exist constants $c_1, c_2 < \infty$ such that

$$\int |K(x,y)|\, d\mu(x) \leq C_1 \qquad \text{for } \mu\text{-a.e. } y,$$

$$\int |K(x,y)|\, d\mu(y) \leq C_2 \qquad \text{for } \mu\text{-a.e. } x.$$

Then the operator $M : L^p \to L^p$ defined by

$$Mf(x) := \int K(x,y) f(y)\, d\mu(y)$$

is well-defined, and $\|M\| \leq C_1^{1/q} C_2^{1/p}$, where $1/p + 1/q = 1$.

Example 1.8. Let X, Y be compact, Hausdorff spaces, and let $\tau : Y \to X$ be continuous. Then the **pullback/composition operator** $\tau^* : C(X) \to C(Y)$ given by $f \mapsto f \circ \tau$ is bounded with $\|\tau^*\| \leq 1$. If $Y \neq \varnothing$, then $\|\tau^*\| = 1$.